ON AN EXACT SOLUTION OF THE EQUATIONS OF MAGNETOHYDRODYNAMICS

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PMM Vol.24, No.2, 1960, pp. 383-386

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(Received 18 August 1959)

1. A paper by Hartmann [1], and also a number of subsequent investigations (for example, [2]), contained studies of the flow of a viscous electrically-conducting fluid between parallel plane walls under the condition that all the parameters are unchanged in the direction of the stream. This restriction, in particular so far as the magnetic field is concerned, can be waived and we then obtain a new exact solution of the equations of magnetohydrodynamics more general than the solutions of [1] and [2].

The fundamental equations are

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\mathbf{V}\right)=-\nabla p^{*}+\eta\Delta\mathbf{V}+\varkappa(\mathbf{H}\nabla)\mathbf{H} \qquad (1.1)$$

$$\frac{\partial \mathbf{H}}{\partial t}+(\mathbf{V}\nabla)\mathbf{H}=(\mathbf{H}\nabla)\mathbf{V}+\lambda\Delta\mathbf{H} \qquad (1.2)$$

$$\operatorname{div}\mathbf{V}=0, \qquad \operatorname{div}\mathbf{H}=0 \qquad (1.3)$$

where

$$p^{*}=p+\mu H^{2}/8\pi, \quad \varkappa=\mu/4\pi, \quad \lambda=c^{2}/4\pi\sigma\mu$$

and the remainder of the notation is conventional. We shall seek a solution of the form

$$v_x=v(y),\ v_y=v_z=0,\ H_x=H_x(x,\ y),\ H_y=H_y(x,\ y),\ H_z=0,\ p^{*}=p^{*}(x,\ y)$$

This solution corresponds to steady flow in the direction of the x-axis in the presence of a certain, as yet undetermined, plane magnetic field. The unknown functions obviously satisfy the following system of equations.

$$\frac{\partial p^*}{\partial x} = \varkappa\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y}\right) + \eta v'' \tag{1.4}$$

$$\frac{\partial p^*}{\partial y} = \varkappa\left(H_x \frac{\partial H_y}{\partial x} + H_y \frac{\partial H_y}{\partial y}\right) \tag{1.5}$$

$$v\frac{\partial H_x}{\partial x} = H_y v' + \lambda \triangle H_x, \qquad v\frac{\partial H_y}{\partial x} = \lambda \triangle H_y \tag{1.6}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \tag{1.7}$$

Let us introduce in these equations the vector potential of the magnetic field A, setting

$$H_x = \partial A / \partial y, \; H_y = -\partial A / \partial x.$$

Then eliminating p^* from (1.4) and (1.5) and integrating Equations (1.6) with respect to x and y respectively, we obtain the nonlinear system

$$\varkappa \frac{D(\triangle A, A)}{D(x, y)} + \eta v''' = 0, \qquad v\frac{\partial A}{\partial x} = \lambda \triangle A + E \tag{1.8}$$

where the constant E is proportional to the z-component of the vector of the electric field and, in the general case, is not equal to zero. If we now eliminate v, taking account of the supplementary condition $\partial v/\partial x = 0$, we obtain two equations which have to be simultaneously satisfied by the potential A

$$\varkappa \frac{D(\triangle A, A)}{D(x, y)} + \eta \frac{\partial^3}{\partial y^3}\left(\frac{\lambda \triangle A + E}{\partial A / \partial x}\right) = 0, \qquad \frac{\partial}{\partial x}\frac{\lambda \triangle A + E}{\partial A / \partial x} = 0 \tag{1.9}$$

It is not difficult to see that these relations are fulfilled if

$$A = -xH_0(y) + \gamma(y) \tag{1.10}$$

where H_0 and γ are determined from the joint system of two equations. Other solutions for A apparently do not exist, since the first equation (1.8),can, by substitution for ΔA from the second equation (1.8), be put into the form

$$v''' - \frac{\varkappa}{\lambda\eta}\left(\frac{\partial A}{\partial x}\right)^2 v' + \frac{\varkappa}{\lambda\eta}\frac{D(\partial A / \partial x, A)}{D(x, y)} v = 0 \tag{1.11}$$

and only (1.10) ensures the independence of these coefficients from x.

The solution (1.10) reduoes to Hartmann's solution [1] when H_0 = constant.

2. Substituting the expression (1.10) into Equation (1.11) and into the second equation (1.8), we obtain

$$v''' - \frac{2\varkappa}{\lambda\eta}\frac{H_0^2}{2}v' - \frac{\varkappa}{\lambda\eta}\left(\frac{H_0^2}{2}\right)' v = 0 \tag{2.1}$$

$$-vH_0 = \lambda(-xH_0'' + \gamma'') + E \tag{2.2}$$

From the latter equation it follows that $H_0''' = 0$, i.e. $H_0 = hy + h_0$ and h and h_0 are constants, which moreover are assumed to be given. Equation (2.1) then has the general integral [3]

$$v = H_0(C_1u_1^2 + C_2u_1u_2 + C_3u_2^2) \tag{2.3}$$

where

$$u_1 = I_{1/4}(mH_0^2/4h), \qquad u_2 = K_{1/4}(mH_0^2/4h), \qquad m = \sqrt{\varkappa/\lambda\eta}$$

Let us now find from Equation (2.2) the quantity γ' which determines the longitudinal component of the field $H_k = -hx + \gamma'$, remembering that $H_0''' = 0$:

$$\gamma' = C_4 - \frac{Ey}{\lambda} - \frac{1}{\lambda}\int vH_0 dy \tag{2.4}$$

To carry out the quadratures in (2.4) we make use of the formulas

$$\begin{gathered}\int z^{1/2} I_{1/4}^2(z)\,dz = z^{3/2}\left[I_{1/4}^2(z) - I_{3/4}^2(z) - \frac{2\sqrt{2}}{\pi} I_{3/4}(z)K_{3/4}(z) - \frac{2}{\pi^2}K_{3/4}^2(z)\right] \\ \int z^{1/2} I_{1/4}(z)K_{1/4}(z)\,dz = z^{3/2}\left[I_{1/4}(z)K_{1/4}(z) - I_{3/4}(z)K_{3/4}(z)\right] \\ \int z^{1/2} K_{1/4}^2(z)\,dz = z^{3/2}\left[K_{1/4}^2(z) - K_{3/4}^2(z)\right]\end{gathered} \tag{2.5}$$

which are derived by the second method of Lommel [4] taking account of the recurrence formulas for $I_\nu(z)$ and $K_\nu(z)$ and the relation

$$K_\nu(z) = \pi\left[I_{-\nu}(z) - I_\nu(z)\right]/2\sin\nu\pi.$$

With the help of (2.5), we find that

$$\begin{gathered}\gamma' = C_4 - \frac{Ey}{\lambda} - \frac{H_0^3}{2h\lambda}\left[C_1\left(u_1^2 - w_1^2 - \frac{2\sqrt{2}}{\pi}w_1w_2 - \frac{2}{\pi^2}w_2^2\right) + \right. \\ \left. + C_2(u_1u_2 - w_1w_2) + C_3(u_2^2 - w_2^2)\right]\end{gathered} \tag{2.6}$$

where

$$w_1 = I_{3/4}(mH_0^2/4h), \qquad w_2 = K_{3/4}(mH_0^2/4h)$$

Formulas (2.3) and (2.6) contain five constants C_i ($i = 1, \ldots, 4$) and E, four of which are determined from the boundary conditions for velocity and the longitudinal component of the field H_x. One constant is related to the pressure gradient. For clarification of this relationship let us substitute $H_y = H_0$, $H_x = -xh + \gamma'$; into Equations (1.4) and (1.5);

we obtain

$$\frac{\partial p^*}{\partial x} = \varkappa(xh^2 + H_0\gamma'' - h\gamma') + \eta v'', \qquad \frac{\partial p^*}{\partial y} = \varkappa h H_0 \tag{2.7}$$

Differentiating the second equation with respect to x we obtain $\partial^2 p^*/\partial x\partial y = 0$, i.e. $\partial p^*/\partial x$ is independent of y. The first equation (2.7) can be presented in the form

$$f(x) = \varkappa(H_0\gamma'' - h\gamma') + \eta v'' \qquad (f(x) = \partial p^*/\partial x - \varkappa x h^2) \tag{2.8}$$

The right-hand side of (2.8) depends only on y, therefore $f(x) = f_0$ = constant. After substituting in (2.8) the expressions we have found for $v(y)$ and $\gamma'(y)$, all the terms in C_1, C_2 and C_3 disappear, and as a result we have

$$f_0 = -\varkappa(Eh_0 + C_4\lambda h) \tag{2.9}$$

The pressure p^* can be calculated from the equations

$$\frac{\partial p^*}{\partial x} = -\varkappa(Eh_0 + C_4\lambda h + xh^2), \qquad \frac{\partial p^*}{\partial y} = \varkappa h(hy + h_0) \tag{2.10}$$

by two quadratures to within an additive constant p_0^* :

$$p^* = p_0^* + \varkappa\left[\frac{h^2y^2}{2} + hh_0 y - \frac{h^2x^2}{2} - (Eh_0 + C_4\lambda h)\,x\right] \tag{2.11}$$

Hence, the purely hydrodynamic pressure p is found as $p^* - \mu H^2/8\pi$.

3. Imposing the appropriate boundary conditions, from the general formulas (2.3), (2.6) and (2.11) we can obtain solutions of problems concerning flow between moving or fixed parallel walls, generalising the results of the papers [1,2] to the case when the transverse component of the field is linearly dependent upon the transverse coordinate. Below we shall consider only the limiting case of flow in a half-space.

We shall take as given the values of the velocity and the longitudinal component of the field on the boundary of the half-space and at infinity, denoting them respectively by

$$U, \quad H_x(0) = \gamma'(0) - hx, \quad v_\infty, \quad H_{x\infty} = \gamma_\infty' - hx \qquad (h > 0,\ h_0 > 0)$$

From the asymptotic behavior [4] of the functions u_i and w_i at large positive values of the argument and $\nu > 0$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}\left[1 - \frac{4\nu^2 - 1}{8z} + o(z^{-4})\right], \quad K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}\left[1 + \frac{4\nu^2 - 1}{8z} + o(z^{-4})\right] \tag{3.1}$$

it follows that

$$H_0u_1^2 \to \infty, \quad H_0u_1u_2 \to 0, \quad H_0u_2^2 \to 0, \quad H_0^3u_1^2 \to \infty, \quad H_0^3u_2^2 \to 0, \quad H_0^3w_1^2 \to \infty$$
$$H_0^3w_2^2 \to 0, \quad H_0^3(u_1u_2 - w_1w_2) \to 0 \quad \text{when} \quad y \to \infty\,(H_0 \to \infty)$$

Therefore, requiring that v and γ' be bounded as $y \to \infty$, we obtain $C_1 = E = 0$, i.e.

$$v = H_0 (C_2 u_1 u_2 + C_3 u_2^2), \qquad \gamma' = C_4 - \frac{H_0^3}{2h\lambda} [C_2 (u_1 u_2 - w_1 w_2) + C_3 (u_2^2 - w_2^2)] \tag{3.2}$$

with $v_\infty = 0$. Calculating C_2, C_3 and C_4 from the boundary conditions, we find eventually that

$$v = \frac{UH_0}{h_0} \frac{[w_{20}^2 - (1+N) u_{20}^2] u_1 u_2 + [(1+N) u_{10} u_{20} - w_{10} w_{20}] u_2^2}{u_{20} w_{20} (u_{10} w_{20} - u_{20} w_{10})} \tag{3.3}$$

$$\gamma' = \gamma_\infty' - \frac{\Gamma H_0^3}{h_0^3} \frac{[w_{20}^2 - (1+N) u_{20}^2] (u_1 u_2 - w_1 w_2) + [(1+N) u_{10} u_{20} - w_{10} w_{20}] (u_2^2 - w_2^2)}{N u_{20} w_{20} (u_{10} w_{20} - u_{20} w_{10})} \tag{3.4}$$

$$(\Gamma = \gamma'(0) - \gamma_\infty', \quad N = 2h\lambda\Gamma / h_0^2 U)$$

Here u_{i0} and w_{i0} are the values of u_i and w_i at the boundary $y = 0$ $(H_0 = h_0)$.

Let us now pass to the limit when $h \to 0$. Making use of the asymptotic formulas (3.1), we obtain the solution in the form

$$v = U - m\lambda\Gamma (1 - e^{-mh_0 y}), \qquad \gamma' = \gamma_\infty' + \Gamma e^{-mh_0 y} \tag{3.5}$$

Here, in the limiting case, v_∞ is different from zero, and

$$\frac{U - v_\infty}{\Gamma} = m\lambda \tag{3.6}$$

If we consider the solution of the same problem with $h = 0$, starting with Equations (2.1) and (2.2), we find first of all that [1,2]

$$v = C_1 + C_2 e^{mh_0 y} + C_3 e^{-mh_0 y}, \gamma' = C_4 - \frac{Ey}{\lambda} - \frac{C_1 h_0 y}{\lambda} - \frac{C_2}{m\lambda} e^{mh_0 y} + \frac{C_3}{m\lambda} e^{-mh_0 y} \tag{3.7}$$

Hence

$$E + C_1 h_0 = 0, \qquad C_2 = 0, \qquad C_1 = v_\infty, \qquad C_4 = \gamma_\infty', \qquad C_3 = U - v_\infty$$

Moreover, from the second of Equations (3.7) it follows that $C_3 = m\lambda\Gamma$.

Accordingly, we again arrive at the solution (3.5) and Equation (3.6) which connect the four boundary conditions of the problem. Equation (3.6) for the particular case $\gamma'(0) = 0$ was presented without derivation in [5], which contained a study of unsteady flow in a half-space and the limiting steady flow.

In a similar manner we can also consider the problem in the case $h < 0$. However, this involves a more precise analysis of the asymptotic properties of cylindrical functions for large negative values of the argument. We notice, moreover, that in the case of different signs of h and

h_0 the quantity H_0 vanishes when $y = -h_0/h$, but the expressions (3.3) and (3.4) still remain bounded.

In conclusion, we mention that the point of the present paper is essentially to find plane magnetic fields whose presence makes possible the plane rectilinear motion of a fluid ($v_x = v(y)$, $v_y = v_z = 0$). As a result we have found a field of the form $\mathbf{H} = (\gamma'(y) - hx)\mathbf{i} + (hy + h_0)\mathbf{j}$. This solution belongs to an extremely broad class of exact solutions of the equations of magnetohydrodynamics which were recently discovered and investigated in general terms in a paper by Lin [6] on the basis of the formal requirements in the structure of the vectors $\mathbf{V}$, $\mathbf{H}$ and ∇p^*.

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Translated by A.H.A.